Conformally Flat Spaces of Locally Constant Connection. II.[1]

B. H. Voorhees

Department of Mathematics, University of Alberta, Edmonton, Alberta, Canada T6G-2G1

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This paper continues the study of conformally flat spaces of locally constant connection to the case in which the global vector field defined by the connection in such spaces is null.

1. INTRODUCTION

This is the second of two papers dealing with conformally flat spaces of locally constant connection. In the first paper (referred to as I) (Voorhees, 1982, this issue), the metric

$$g_{ij} = e^{2\Phi}\eta_{ij}$$

$$\phi = (\alpha \cdot x) \tag{1}$$

was used where $\eta_{ij} = \text{diag}(1,1,1,-1)$ and $(\alpha \cdot x) = \alpha^1 x^1 + \alpha^2 x^2 + \alpha^3 x^3 - \alpha^4 x^4$. Notation follows I. In I it was shown that a nonnegative energy condition was satisfied if and only if α^i was timelike or null. Paper I focused on the case where α^i was timelike. This paper considers the case where α^i is null and future pointing.

The geodesic equations for the metric (1) can be put into the form

$$\frac{dx^i}{dt} = u^i$$

$$\frac{du^i}{dt} = |u|^2\alpha^i - 2(\alpha \cdot u)u^i \tag{2}$$

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where $|\cdot|^2$ and $(\cdot)$ indicate products taken with respect to the Minkowski metric η_{ij}. The general solution of these equations was given in I as

$$u^i(t)=\frac{u^i(0)+|u(0)|^2 t\alpha^i}{1+2(\alpha\cdot u(0))t+|\alpha|^2|u(0)|^2t^2} \tag{3}$$

where t is an affine parameter. If α^i is null $|\alpha|^2=0$ and (3) becomes

$$u^i(t)=\frac{u^i(0)+|u(0)|^2 t\alpha^i}{1+2(\alpha\cdot u(0))t} \tag{4}$$

As in I it is useful to consider the cases in which $u^i(0)$ is spacelike, null, and timelike separately.

Case I: $u^i(0)$ spacelike. Normalizing so that $|u(0)|=1$ we see that there are two possibilities depending on whether or not $(\alpha\cdot u(0))=0$. If this is the case

$$u^i(t)=u^i(0)+t\alpha^i \tag{5}$$

and geodesics are complete. If $(\alpha\cdot u(0))\neq 0$ the denominator of (4) becomes zero for

$$t_*=-\frac{1}{2(\alpha\cdot u(0))} \tag{6}$$

Writing $\alpha^i=(\boldsymbol{\alpha},\alpha^4)$, $u^i(0)=(\mathbf{u}(0),u^4(0))$ the denominator in this expression may be written as

$$2|\boldsymbol{\alpha}|\,|\mathbf{u}(0)|\left[\cos\theta-\frac{u^4(0)}{|\mathbf{u}(0)|}\right] \tag{7}$$

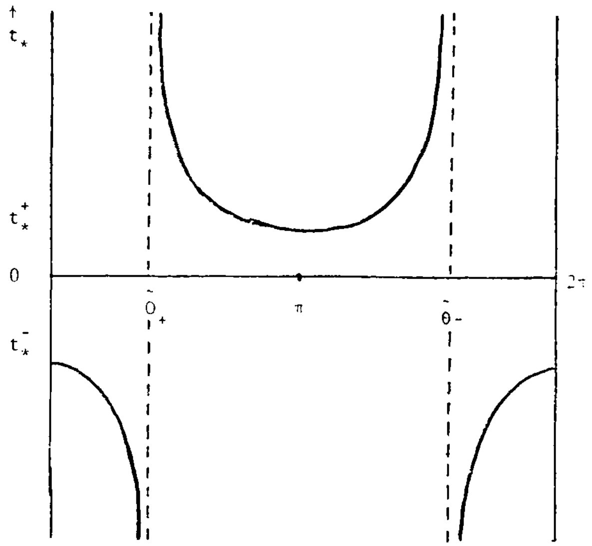
If $u^i(0)$ is spacelike $u^4(0)<|\mathbf{u}(0)|$ and t_* can be plotted against θ, the angle between $\boldsymbol{\alpha}$ and $\mathbf{u}(0)$. This is done in Figure 1. For the particular values

$$\tilde{\theta}_{\pm}=\pm\cos^{-1}\left[\frac{u^4(0)}{|\mathbf{u}(0)|}\right] \tag{8}$$

t_* is undefined. These particular geodesics are complete. If $-\tilde{\theta}<\theta<\tilde{\theta}_+$ the geodesics are complete for $t\to\infty$. They are complete for $t\to-\infty$ otherwise.

Case II: $u^i(0)$ null. Now (4) becomes

$$u^i(t)=\frac{u^i(0)}{1+2(\alpha\cdot u(0))t} \tag{9}$$



$$t_*^+ = \left[2|\vec{\alpha}|\ |\vec{u}(0)|\left(1+\frac{u^4(0)}{|\vec{u}(0)|}\right)\right]^{-1}$$

$$t_*^- = -\left[2|\vec{\alpha}|\ |\vec{u}(0)|\left(1-\frac{u^4(0)}{|\vec{u}(0)|}\right)\right]^{-1}$$

Fig. 1. Values of affine parameters at which spacelike geodesics are incomplete.

If $u^i(0)$ is proportional to α^i then $(\alpha \cdot u(0)) = 0$ and $u^i(t) = u^i(0)$. Otherwise, the geodesics are incomplete at the value t_* of (6). However, $u^i(0)$ is now null and so $u^4(0) = |\mathbf{u}(0)|$. Figure 2 shows the regions of the null cone at the initial position covered by applying the inverse exponential map to null geodesics through that position. Note that these spaces admit a global family of null hypersurfaces with null generator α^i.

Case III: $u^i(0)$ timelike. Again these geodesics are incomplete at t_* of (6). Now, however, the denominator of (6) is always negative and hence these geodesics are always incomplete in the direction of $u^i(0)$.

Integration of $u^i = dx^i/dt$ yields

$$x^i(t) = \frac{u^i(0)}{2(\alpha \cdot u(0))} \ln[1 + 2(\alpha \cdot u(0))t] + \alpha^i \left\{ \frac{|u(0)|^2 t}{2(\alpha \cdot u(0))} \quad \frac{1}{4(\alpha \cdot u(0))^2} \ln[1 + 2(\alpha \cdot u(0))t] \right\} \tag{10}$$

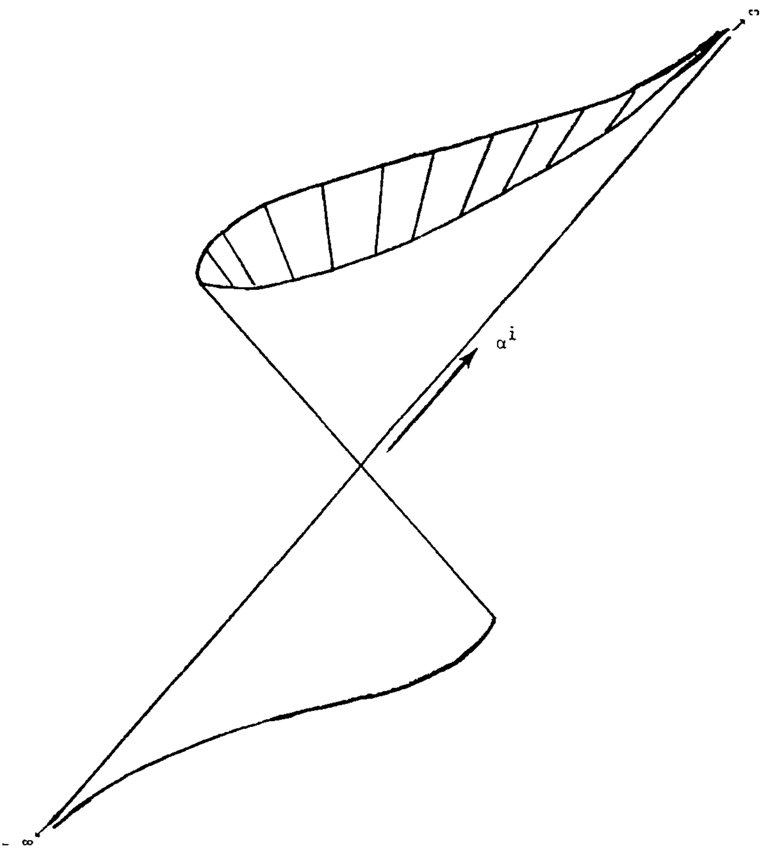


Fig. 2. Maximum extent of null geodesics.

so long as $(\alpha \cdot u(0)) \neq 0$. Thus at t_* the coordinates values $x^i(t)$ diverge for finite values of the affine parameter.

2. GEOMETRY

From the appendix of I the Riemann, Ricci, and scalar curvatures for a space of locally constant connection with α^i null are given by

$$R^{ij}_{kl} = e^{-4\theta}\left[\alpha^i\left(\delta^j_l \alpha_k - \delta^j_k \alpha_l\right) + \alpha^j\left(\delta^i_k \alpha_l - \delta^i_l \alpha_k\right)\right]$$

$$R^i_j = G^i_j = 2e^{-4\theta}\alpha^i\alpha_j$$

$$R = 0 \qquad (11)$$

where $\phi = \eta_{ij}\alpha^i x^j$. The source of the exponential factors lies in use of the conformal metric g^{ij} to raise indices, and the identity $\phi_{,i} = e^{-2\phi}\alpha_i$.

Using the classification of Ludwig and Scanlan (1971) the Ricci tensor is of type A_{3+} and thus its eigenvectors span a family of null hyperplanes which foliate the space-time. In addition, all eigenvalues are zero.

We can also inquire into eigenforms of the Riemann tensor:

> *Theorem.* The Riemann tensor of a conformally flat space of locally constant connection with α^i null has a four-parameter family of eigenforms with degenerate eigenvalue zero. There are no other eigenforms.

Proof. Let $(\alpha^i, \beta^i, m^i, \overline{m}^i)$ be a null tetrad with normalization

$$\alpha_i \beta^i = -m_i \overline{m}^i = 1$$

and all other products zero. Any 2-form ω^{kl} can be written in the form

$$\omega^{kl} = A\alpha^{[k}\beta^{l]} + B\alpha^{[k}m^{l]} + \overline{B}\alpha^{[k}\overline{m}^{l]} + C\beta^{[k}m^{l]} + \overline{C}\beta^{[k}\overline{m}^{l]} + Dm^{[k}\overline{m}^{l]} \tag{12}$$

with appropriate choice of the coefficients. Transvecting with the R^{ij}_{kl} of (11) yields

$$R^{ij}_{kl}\omega^{kl} = 2C\alpha^{[i}m^{j]} + 2\overline{C}\alpha^{[i}\overline{m}^{j]} \tag{13}$$

We see by inspection of (12) and (13) that ω^{kl} solves the eigenvalue equation if and only if $C = \overline{C} = 0$. This leaves a four-parameter family of solutions with eigenvalue zero. ∎

3. SUMMARY

The main characteristic of these space-times appears to be the existance of a one-parameter family of null hypersurfaces with null generator α^i which foliate the manifold. Inspection of the Einstein tensor indicates that the source of these space-times is null radiation in the α^i direction.

In I it was shown that the space-times with α^i timelike were particular Robinson–Walker universes. If α^i is null, however, the space-time cannot be Robinson–Walker since it is not isotropic—the spatial projection of α^i defines a preferred direction (this point was brought to my attention by Werner Israel).

If we introduce an orthonormal frame (x^i, y^i, z^i, t^i) and in terms of this define a null frame

$$l^i = \frac{1}{\sqrt{2}}(t^i - z^i)$$

$$n^i = \frac{1}{\sqrt{2}}(t^i + z^i)$$

$$m^i = \frac{1}{\sqrt{2}}(x^i + iy^i)$$

$$\overline{m}^i = \frac{1}{\sqrt{2}}(x^i - iy^i) \tag{14}$$

then, assuming our frame vectors are chosen so that

$$\alpha^i = \frac{1}{\sqrt{2}}(Al^i + Bn^i), \tag{15}$$

the nonzero components of the curvature tensor are

$$\Phi_{00} = -B^2 e^{-4\phi}$$

$$\Phi_{11} = -\tfrac{1}{2}ABe^{-4\phi}$$

$$\Phi_{22} = -A^2 e^{-4\phi} \tag{16}$$

together with the scalar curvature which is zero only if α^i is null. This occurs in the limit $B \to 0$. If α^i is timelike we can normalize so that $\alpha_i \alpha^i = -1$, implying $B = -1/A$. In the limit $A \to \infty$, corresponding to a singular boost in the (l^i, n^i) plane, (16) indicates that the Φ's approach those for α^i null, but Φ_{11} does not go to zero. The quantity which becomes exactly the Φ's with α^i null is $(1/A^2)\Phi$.

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